EE 508
Lecture 2

Filter Design Process

Review from Last Time

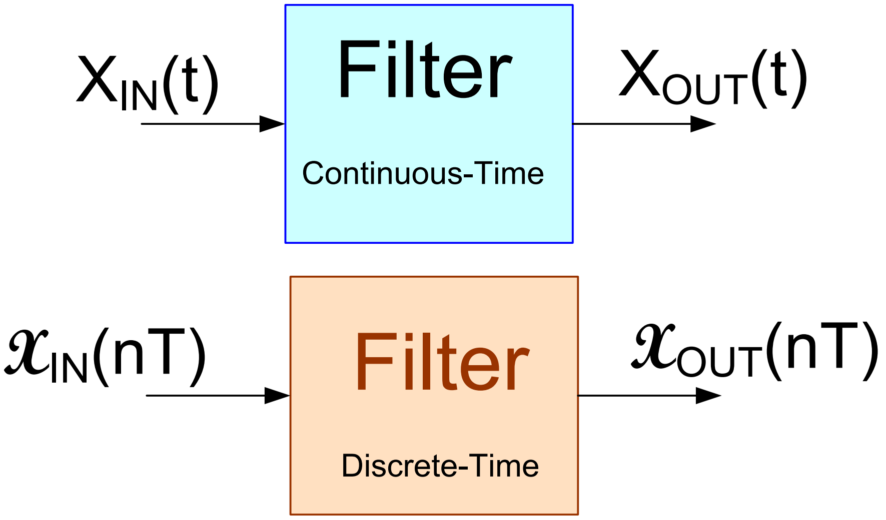
What is a filter?

Conceptual definition:

A filter is an amplifier or a system that has a frequency dependent gain

Note:

Implicit assumption is made in this definition that the system is linear. In this course, will restrict focus to filters that are ideally linear



$X_{IN}(t)$ → Filter (Continuous-Time) → $X_{OUT}(t)$

$\mathcal{X}_{IN}(nT)$ → Filter (Discrete-Time) → $\mathcal{X}_{OUT}(nT)$

Filters can be continuous-time or discrete-time

Review from Last Time

Filter design field has received considerable attention by engineers for about 8 decades

- Passive RLC
- Vacuum Tube Op Amp RC
- Active Filters (Integrated op amps, R,C)
- Digital Implementation (ADC,DAC,DSP)
- Integrated Filters (SC)
- Integrated Filters (Continuous-time and SC)

Review from Last Time

Filter: Amplifier or system that has a frequency-dependent gain

- Filters are ideally linear devices
  - Analog filters characterized by linear differential equations
  - Digital filters characterized by linear difference equations

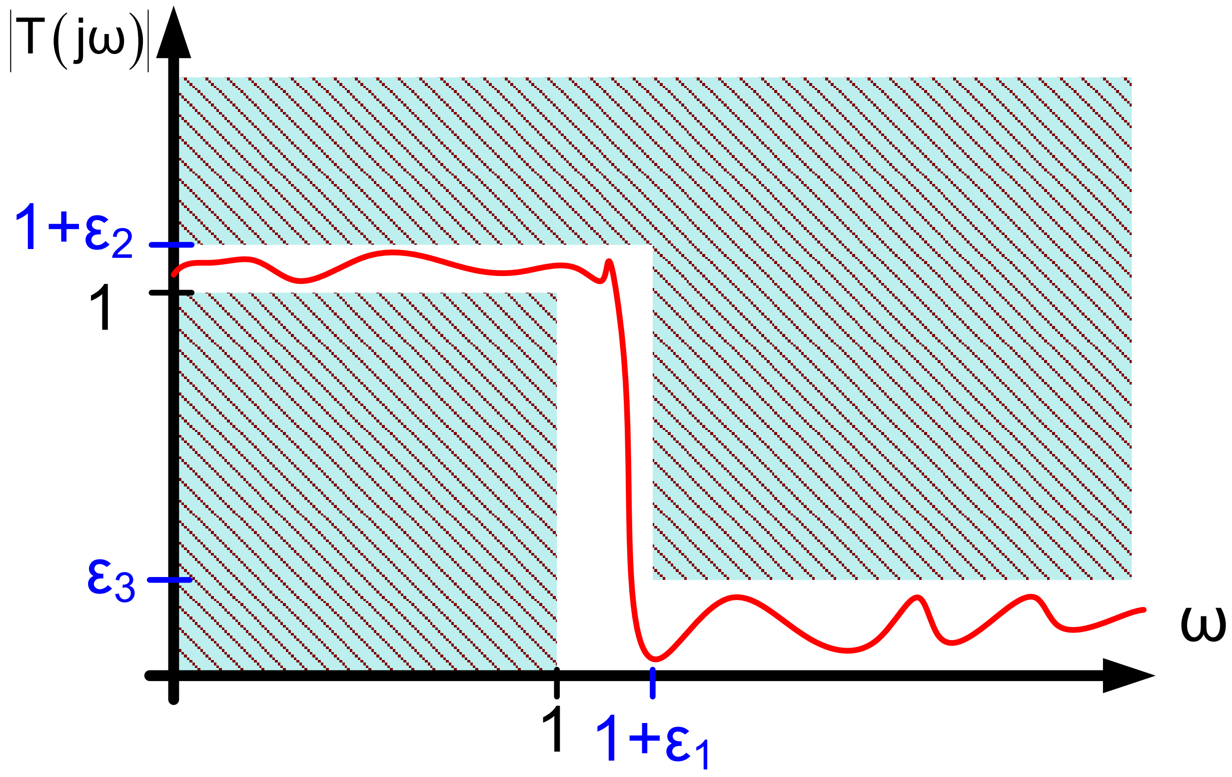
- Characteristics usually expressed as either frequency response or time domain response

- Transfer functions of filters with finite number of lumped elements (analog) or a finite number of additions (digital) are rational fractions with real coefficients

- Transfer functions of any realizable filter (finite elements or additions) have no discontinuities in either the magnitude or phase response

Review from Last Time

Any circuit that has a transfer function that does not enter the forbidden region is an acceptable solution from a performance viewpoint



Review from Last Time

- Minor changes in specifications can have significant impact on cost and effort for implementing a filter
- Work closely with the filter user to determine what filter specifications are really needed

Observations about Filter Transfer Functions

$$T(s)=\frac{\sum_{i=1}^{m} a_i s^i}{\sum_{i=1}^{n} b_i s^i}=\frac{N(s)}{D(s)}$$

$$H(z)=\frac{\sum_{i=1}^{m} a_i z^i}{\sum_{i=1}^{n} b_i z^i}=\frac{N(z)}{D(z)}$$

Transfer functions characterize the steady-state response of a filter and are unaffected by the initial conditions

Transfer functions of any filter are rational fractions with real coefficients

Filters always operate in the time domain but are often characterized in the frequency domain

T(s) or H(z) can be obtained by taking the Laplace Transform or z-transform of the differential equation or difference equation describing the operation of the filtger and then solving for ratio of output to input

Often easier ways to obtain T(s) or H(z)

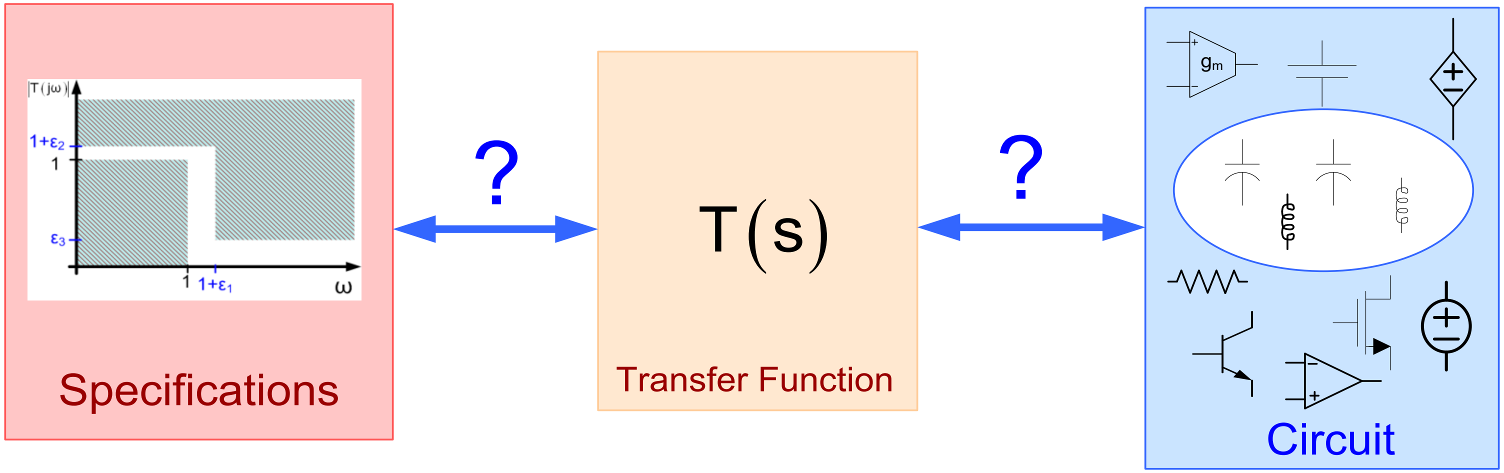
Is there a systematic way to design filters?

Observations:

- All analog filter circuits with a finite number of lumped elements have a transfer function that is a rational fraction in s
- All digital filters have a transfer function that is a rational fraction in z
- Most of the characteristics of a filter are determined by the transfer function

Is there a systematic way to design filters?

(Consider continuous-time first)



Specifications

$T(s)$

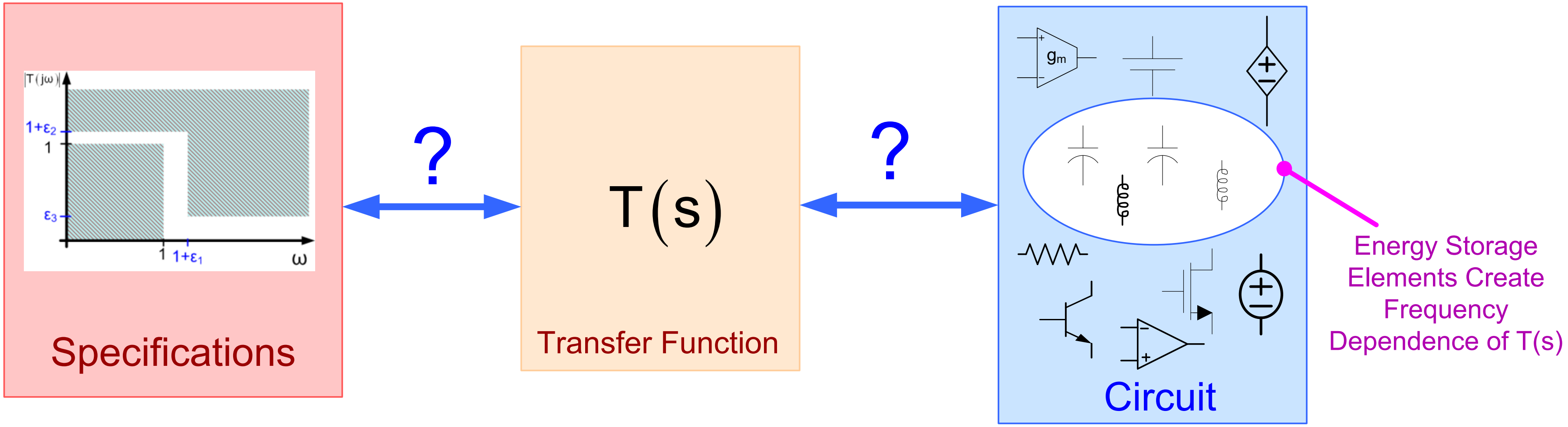
Transfer Function

Circuit

Filter Design Strategy: Use the transfer function as an intermediate step between the Specifications and Circuit Implementation

Is there a systematic way to design filters?

(Consider continuous-time first)



Specifications

T(s)

Transfer Function

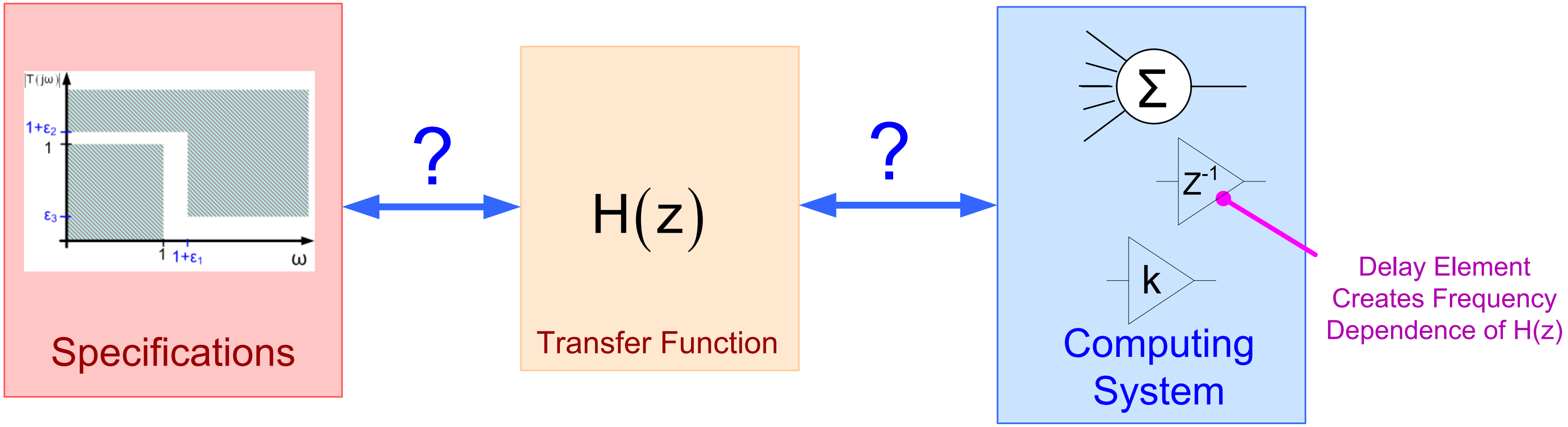
Circuit

Energy Storage Elements Create Frequency Dependence of T(s)

Filter Design Strategy: Use the transfer function as an intermediate step between the Specifications and Circuit Implementation

Is there a systematic way to design filters?

(Consider discrete-time domain)



Specifications

H(z)

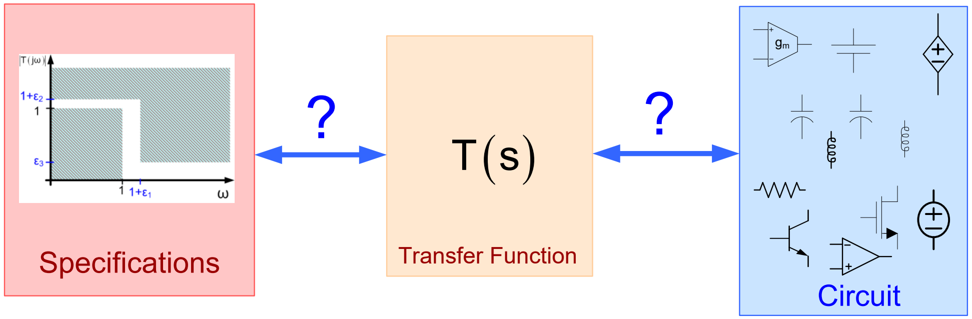
Transfer Function

Computing System

Delay Element
Creates Frequency
Dependence of H(z)

Filter Design Strategy: Use the transfer function as an intermediate step between the Specifications and Circuit Implementation

Filter Design Process



Filter Design Strategy: Use the transfer function as an intermediate step between the Specifications and Circuit Implementation

Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation

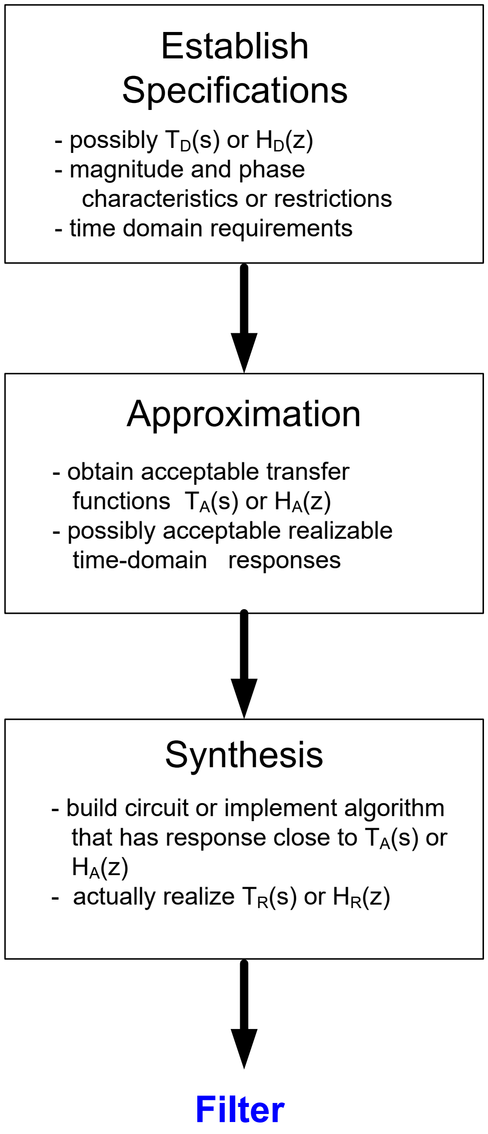
- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Synthesis

- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Filter

Filter Design Process



Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Must understand the real performance requirements

Approximation

- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Obtain an acceptable approximating function ($T_A(s)$ or $H_A(z)$)

Synthesis

- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Design (synthesize) a practical circuit or system that has a transfer function close to the acceptable approximating function

Filter

Filter Design Process

Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation

- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Synthesis

- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Filter

Must understand the real performance requirements

- **Many acceptable specifications for a given application**
- **Some much better than others**
- **But often difficult to obtain even one that is useful**

Obtain an acceptable approximating function ($T_A(s)$ or $H_A(z)$)

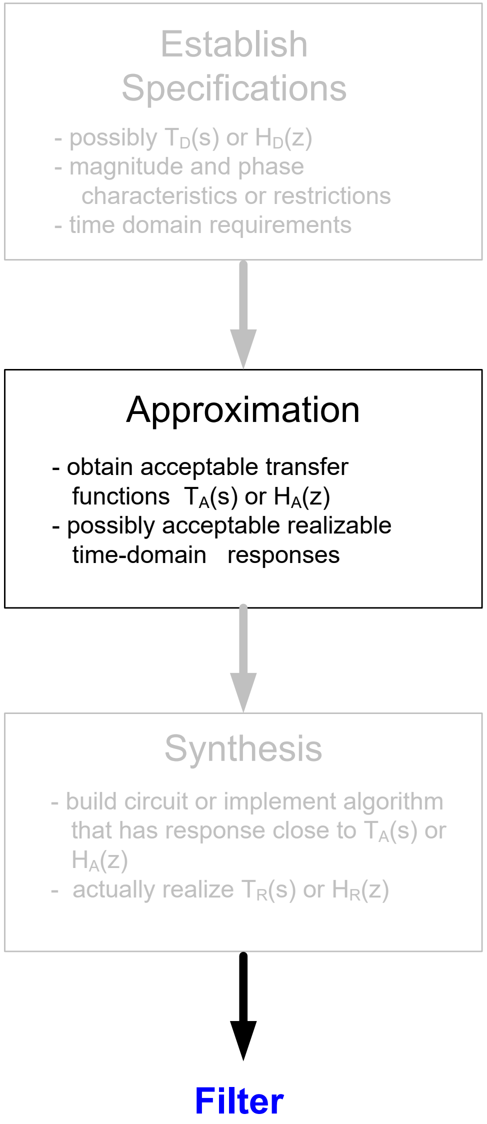
- **Many acceptable approximating functions for a given specification**
- **Some much better than others**
- **But often difficult to obtain even one!**

Design (synthesize) a practical circuit or system that has a transfer function close to the acceptable approximating function

- **Many acceptable circuits or systems for a given approximating function**
  Some much better than others
- **But often difficult to obtain even one!**

Important to make good decisions at each step in the filter design process because poor decisions will not be absolved in subsequent steps

Filter Design Process



Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation

- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

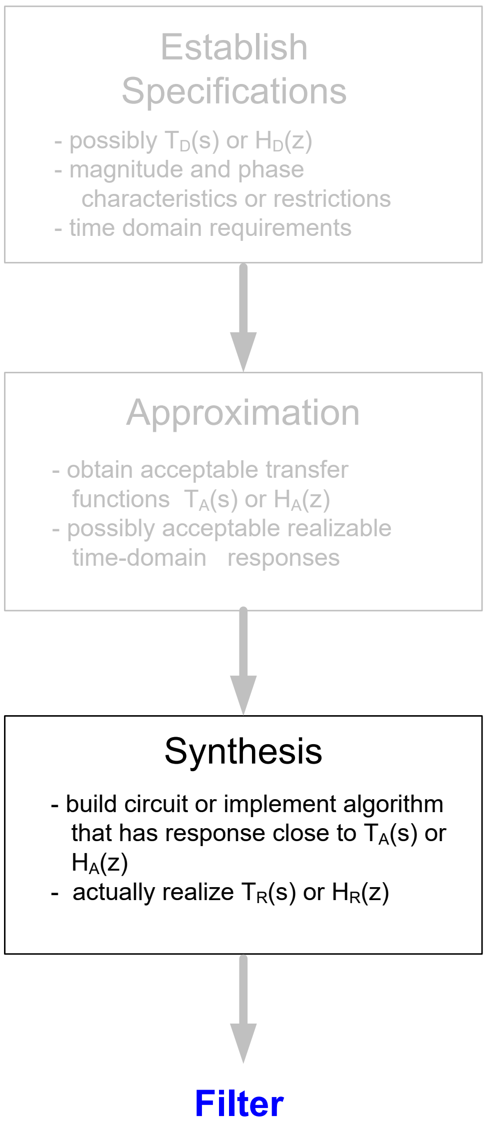
Synthesis

- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Filter

- Order of approximating function directly affects cost of implementation
- Number of energy storage elements in circuit is equal to the order of T(s) (neglecting energy storage element loops)
- High Q poles and zeros adversely affect cost (because component tolerances become tight)
- Cost of implementation (synthesis) is essentially independent of the quality of the approximation if the order is fixed
- Major effort over several decades was focused on the approximation problem

Filter Design Process



Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation

- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Synthesis

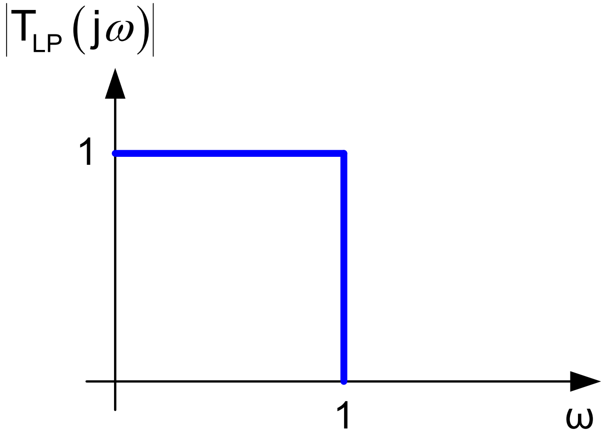
- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Filter

Some realizations are much better than others

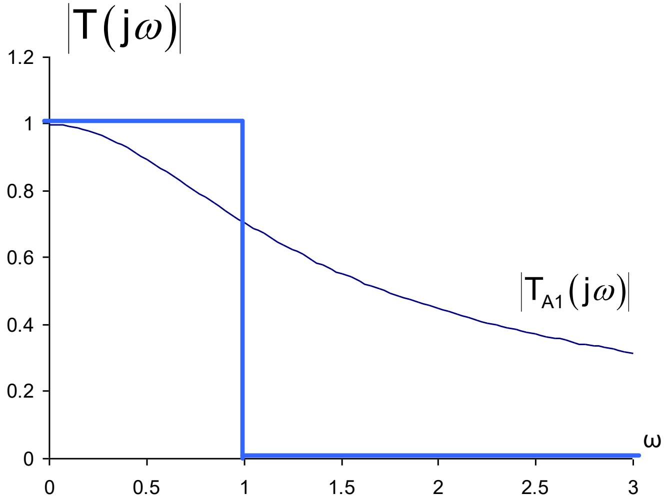
- Cost
- Sensitivity
- Tunability
- Parasitic Effects
- Linearity
- Area
- **Major effort over several decades focused on synthesis problem**

Example:
Design a filter that approximates the ideal lowpass filter



Desired filter magnitude response

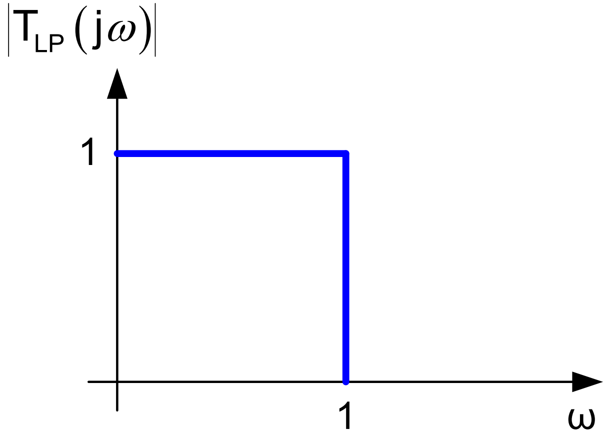
(No phase constraints)



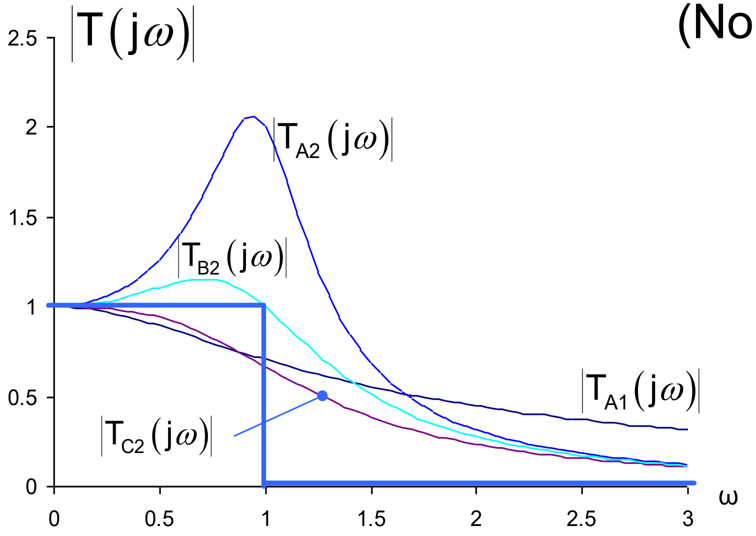
$$T_{A1} = \frac{1}{s+1}$$

One approximating function

Example:
Design a filter that approximates the ideal lowpass filter



Desired filter magnitude response
(No phase constraints)



$$T_{A1}=\frac{1}{s+1}$$

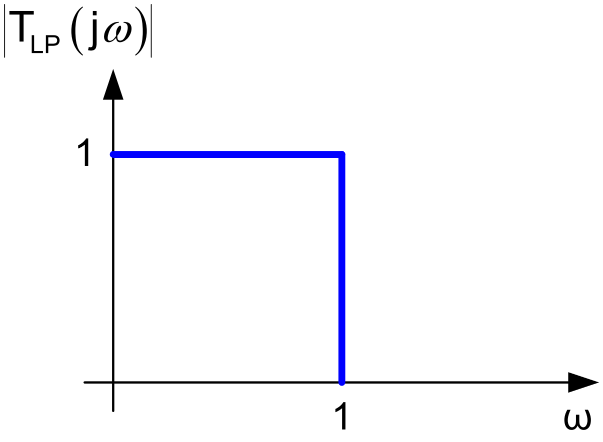
$$T_{A2}=\frac{1}{s^2+0.5s+1}$$

$$T_{B2}=\frac{1}{s^2+s+1}$$

$$T_{C2}=\frac{1}{s^2+1.5s+1}$$

Some additional approximating functions

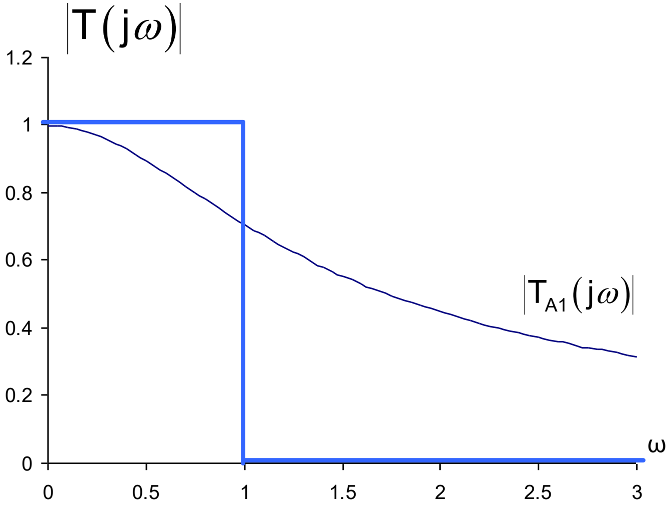
Example:
Design a filter that approximates the ideal lowpass filter

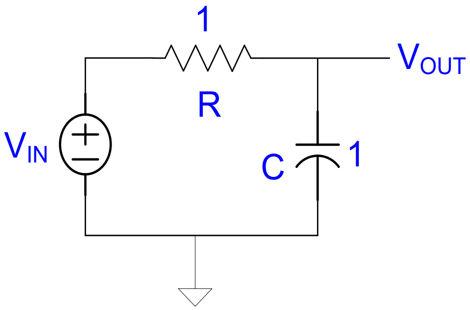


Desired filter magnitude response

(No phase constraints)

$$T_{A1} = \frac{1}{s+1}$$



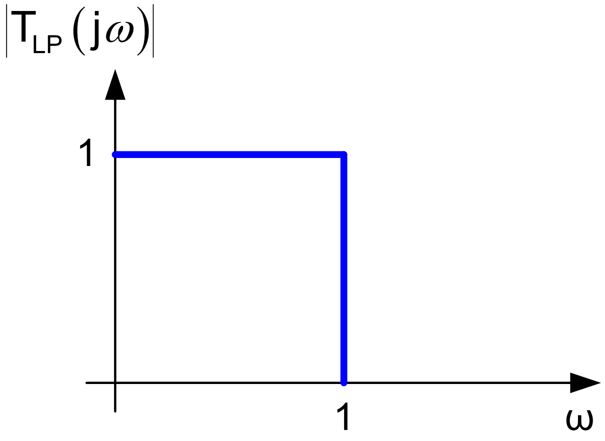


A circuit that realize T_{A1}

$$T(s) = \frac{1}{1+RCs}$$

But not practical because C is too large!

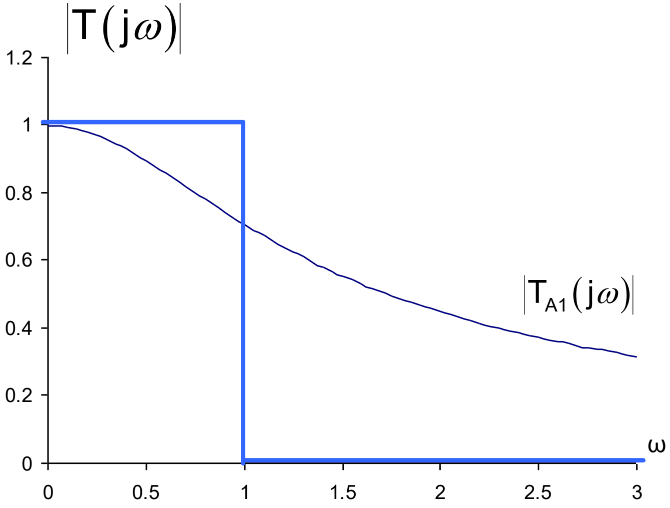
Example:
Design a filter that approximates the ideal lowpass filter

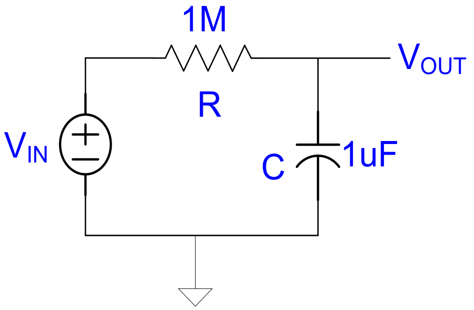


Desired filter magnitude response

(No phase constraints)

$$T_{A1}=\frac{1}{s+1}$$



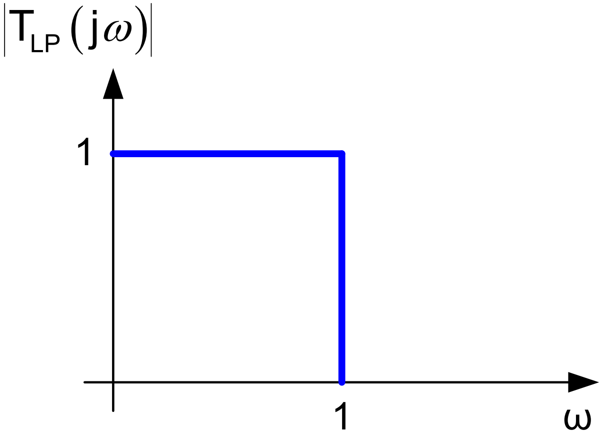


A circuit that realize T_{A1}

$$T(s)=\frac{1}{1+RCs}$$

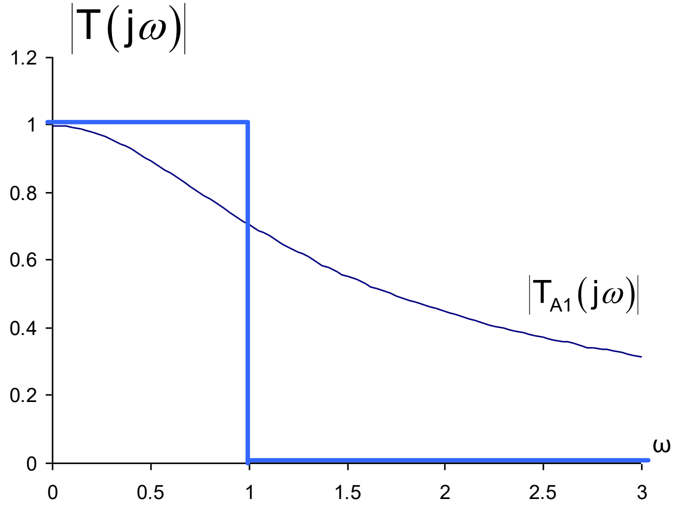
More practical (C must not be electrolytic)!

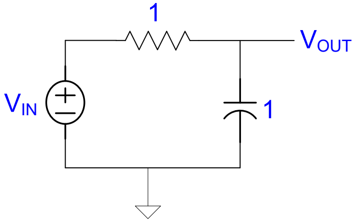
Example:
Design a filter that approximates the ideal lowpass filter



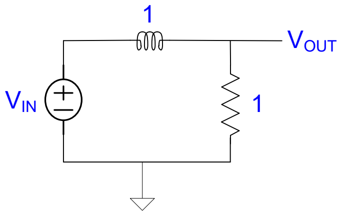
Desired filter magnitude response

$$T_{A1}=\frac{1}{s+1}$$

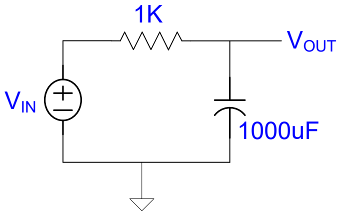




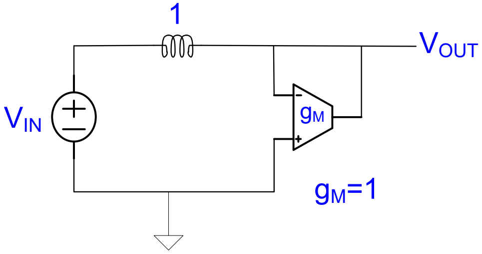
$$T(s)=\frac{1}{1+RCs}$$



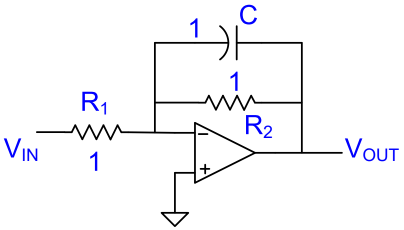
$$T(s)=\frac{1}{1+\frac{L}{R}s}$$



$$T(s)=\frac{1}{1+RCs}$$



$$T(s)=\frac{1}{1+g_mLs}$$

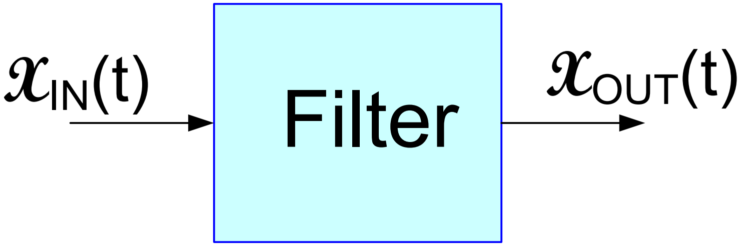


$$T(s)=\frac{-{R_2}/{R_1}}{1+R_2Cs}$$

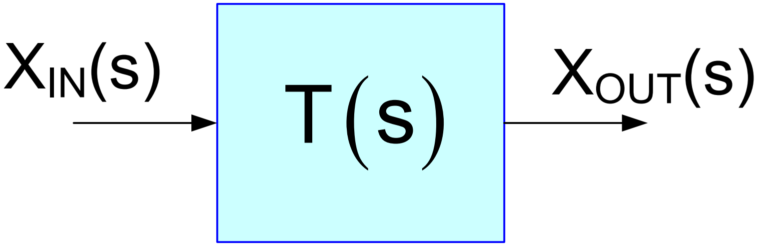
Some additional circuits that realize T_{A1}

Time Domain and Frequency Domain Characterization

Filters always operate in the time domain



Filters often characterized/designed in the frequency domain



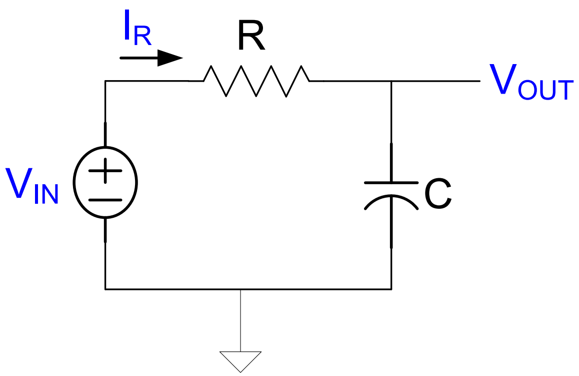
$$T(s) = \frac{X_{OUT}(s)}{X_{IN}(s)} \longrightarrow T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \qquad T(s) = \frac{\mathcal{L}(\mathscr{X}_{OUT}(t))}{\mathcal{L}(\mathscr{X}_{IN}(t))} \longrightarrow ?$$

$$m \le n$$

Time Domain and Frequency Domain Characterization

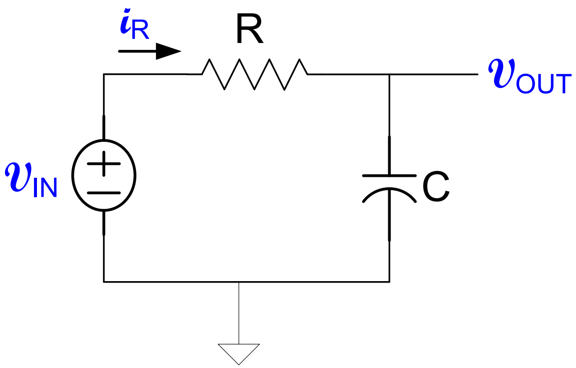
Example:

Frequency Domain



$$\left.\begin{aligned} I_R &= \frac{V_{IN}-V_{OUT}}{R} \\ I_R \bullet \frac{1}{sC} &= V_{OUT}\end{aligned}\right\} \rightarrow \quad T(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} = \frac{1}{1+RCs} = \frac{1}{1+b_1 s}$$

Time Domain



$$\left.\begin{aligned} i_R &= \frac{v_{IN}-v_{OUT}}{R} \\ i_R &= C\frac{dv_{OUT}}{dt}\end{aligned}\right\} \rightarrow \quad \frac{dv_{OUT}}{dt} = \left(\frac{1}{RC}\right)v_{IN} - \left(\frac{1}{RC}\right)v_{OUT}$$

$$\frac{dv_{OUT}}{dt} = \left(\frac{1}{b_1}\right)v_{IN} - \left(\frac{1}{b_1}\right)v_{OUT}$$

Differential Equation

Taking the Laplace transform of the differential equation, we obtain

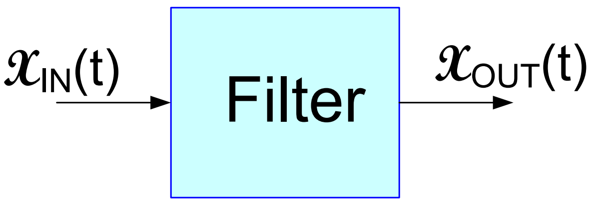
$$\left.\begin{aligned} \mathcal{L}\left(\frac{dv_{OUT}}{dt}\right) &= \left(\frac{1}{b_1}\right)\mathcal{L}(v_{IN}) - \left(\frac{1}{b_1}\right)\mathcal{L}(v_{OUT}) \\ sV_{OUT} &= \left(\frac{1}{b_1}\right)V_{IN} - \left(\frac{1}{b_1}\right)V_{OUT}\end{aligned}\right\}$$

$$T(s) = \frac{V_{OUT}}{V_{IN}} = \frac{1}{1+b_1 s}$$

Time Domain and Frequency Domain Characterization

Generalizing from the previous example:

Time Domain



$x_{\text{IN}}(t)$ → Filter → $x_{\text{OUT}}(t)$

Elements in filter are {R's, C's,L's, indep sources, dep sources}

Assume n energy storage elements and no energy storage element loops in the circuit

The relationship between $x_{\text{OUT}}(t)$ and $x_{\text{IN}}(t)$ can always be expressed by a single time-domain differential equation as

$$\frac{d^n v_{\text{OUT}}}{dt^n} = \sum_{k=0}^{m} \alpha_k \frac{d^k v_{\text{IN}}}{dt^k} - \sum_{k=0}^{n-1} \beta_k \frac{d^k v_{\text{OUT}}}{dt^k}$$

where the α_k and β_k are constants dependent on the values of the circuit elements

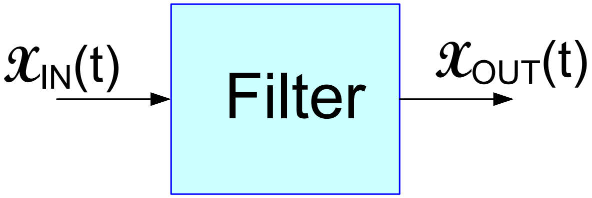
Taking the Laplace transform of this differential equation, we obtain

$$s^n V_{\text{OUT}} = \sum_{k=0}^{m} \alpha_k s^k V_{\text{IN}} - \sum_{k=0}^{n} \beta_k s^k V_{\text{OUT}}$$

Time Domain and Frequency Domain Characterization

Generalizing from the previous example:

Time Domain



$$s^n V_{OUT} = \sum_{k=0}^{m} \alpha_k s^k V_{IN} - \sum_{k=0}^{n-1} \beta_k s^k V_{OUT}$$

If we define $\beta_n=1$, this can be rewritten as

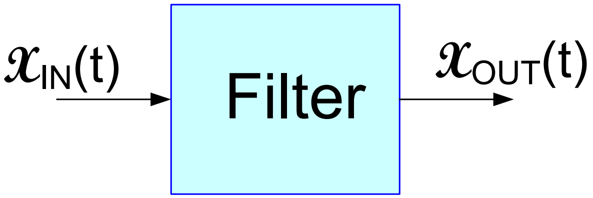
$$\left(\sum_{k=0}^{n} \beta_k s^k\right) V_{OUT} = \sum_{k=0}^{m} \alpha_k s^k V_{IN}$$

Thus, the transfer function can be written as

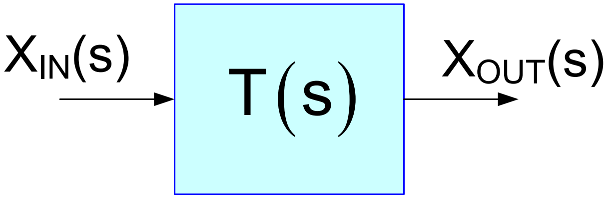
$$T(s) = \frac{V_{OUT}}{V_{IN}} = \frac{\sum_{k=0}^{m} \alpha_k s^k}{\sum_{k=0}^{n} \beta_k s^k}$$

Time Domain and Frequency Domain Characterization

Time Domain



Frequency Domain



$$\frac{d^n v_{OUT}}{dt^n} = \sum_{k=0}^{m} \alpha_k \frac{d^k v_{IN}}{dt^k} - \sum_{k=0}^{n-1} \beta_k \frac{d^k v_{OUT}}{dt^k}$$

$$T(s) = \frac{\sum_{k=0}^{m} \alpha_k s^k}{\sum_{k=0}^{n} \beta_k s^k}$$

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

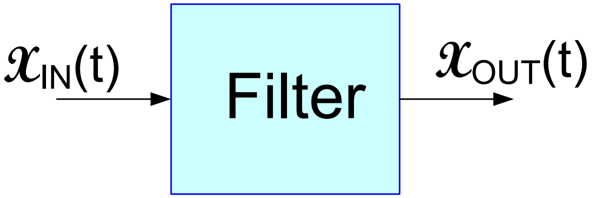
How do the α_k and β_k parameters relate to the a_k and b_k paramaters?

If we normalize the frequency-domain solution so that $b_n=1$, then

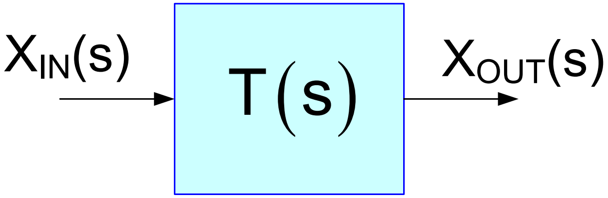
$\alpha_k=a_k$ and $\beta_k=b_k$ for all k

Time Domain and Frequency Domain Characterization

Time Domain



Frequency Domain



$$\frac{d^n v_{OUT}}{dt^n} = \sum_{k=0}^{m} \alpha_k \frac{d^k v_{IN}}{dt^k} - \sum_{k=0}^{n-1} \beta_k \frac{d^k v_{OUT}}{dt^k}$$

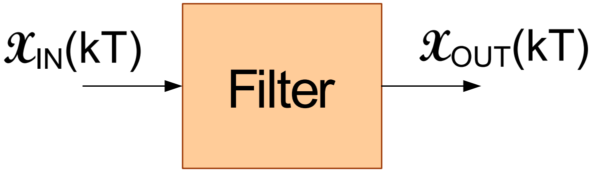
$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

Thus, the time-domain characterization of a filter which can be expressed as a single differential equation can be obtained directly from the transfer function T(s) obtained from a frequency-domain analysis of the circuit

This differential equation does not contain any initial condition information

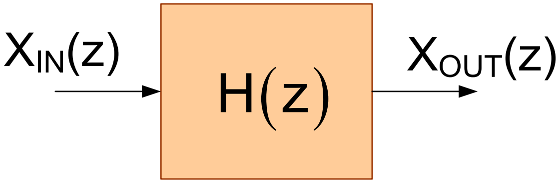
Time Domain and Frequency Domain Characterization

Time Domain



$x_{IN}(kT)$ → Filter → $x_{OUT}(kT)$

Frequency Domain



$X_{IN}(z)$ → $H(z)$ → $X_{OUT}(z)$

$$v_{OUT}(nT) = \sum_{k=0}^{m} \alpha_k v_{IN}((n-k)T) - \sum_{k=1}^{n-1} \beta_k v_{OUT}((n-k)T)$$

If we define $\beta_0=1$ and take the z-transform of the difference equation, obtain

$$H(z) = \frac{\sum_{k=0}^{m} \alpha_k z^{-k}}{\sum_{k=0}^{n} \beta_k z^{-k}}$$

$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i}$$

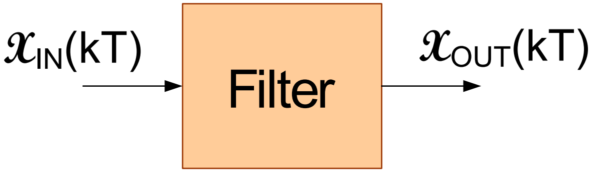
How do the α_k and β_k parameters relate to the a_k and b_k paramaters?

If we normalize the frequency-domain solution so that $b_n=1$ and assume $n \geq m$ then

$a_k = \alpha_{n-k}$ and $b_k = \beta_{n-k}$ for all k

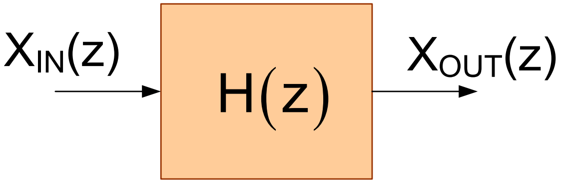
Time Domain and Frequency Domain Characterization

Time Domain



$x_{IN}(kT)$ Filter $x_{OUT}(kT)$

Frequency Domain



$X_{IN}(z)$ $H(z)$ $X_{OUT}(z)$

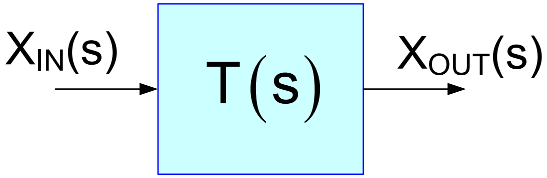
$$v_{OUT}(nT) = \sum_{k=0}^{m} \alpha_k v_{IN}\left(\left(n-k\right)T\right) - \sum_{k=1}^{n-1} \beta_k v_{OUT}\left(\left(n-k\right)T\right)$$

$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i}$$

Thus, the time-domain characterization of a filter which can be expressed as a single difference equation can be obtained directly from the transfer function H(z) obtained from a frequency-domain analysis of the circuit

This difference equation does not contain any initial condition information

Filter Concepts and Terminology



$$T(s)=\frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}=\frac{N(s)}{D(s)}$$

- A polynomial in **s** is said to be “integer monic” if the coefficient of the highest-order term is 1

- If D(s) is integer monic, then N(s) and D(s) for any filter are unique

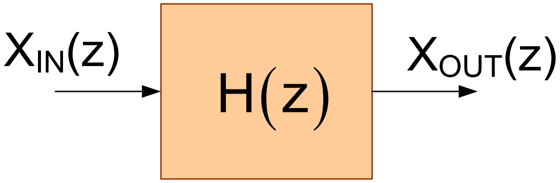
- If D(s) is integer monic, then the a_k and b_k terms are unique

- The roots of N(s) are termed the zeros of the transfer function

- The roots of D(s) are termed the poles of the transfer function

- If N(s) and D(s) are of orders m and n respectively, then there are m zeros and n poles in T(s)

Filter Concepts and Terminology



$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{N(z)}{D(z)}$$

- A polynomial in **z** is said to be “integer monic” if the coefficient of the highest-order term is 1
- If D(z) is integer monic, then N(z) and D(z) are unique
- If D(z) is integer monic, then the a_k and b_k terms are unique
- The roots of N(z) are termed the zeros of the transfer function
- The roots of D(z) are termed the poles of the transfer function
- If N(z) and D(z) are of orders m and n respectively, then there are m zeros and n poles in H(z)

Stay Safe and Stay Healthy !

End of Lecture 2